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Kok, L.P.; Roo, M. de

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AMBIGUOUS SETS OF PARTIAL-WAVE AMPLITUDES CAN INTERSECT

L.P. KOK *
The Weizmann Institute of Science, Rehovot, Israel

M. de ROO
Service de Physique Théorique, CEN-Saclay, B.P. no. 2-91, Gif sur Yvette, France

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We discuss continuation in energy as a method to resolve ambiguities in phase-shift analysis. We show that continuity in energy is not in all cases sufficient to resolve ambiguities, and we give examples of such cases for both spin 0-spin 0 and spin 0-spin $\frac{1}{2}$ scattering.

1. Introduction

The possible ambiguities in phase-shift analysis have recently attracted considerable attention, from both the theoretical and experimental points of view. Of course, it is equally, if not more interesting, to find methods to resolve these ambiguities. The question, whether or not such ambiguities in analysis of spin 0-spin $\frac{1}{2}$ scattering can always be resolved by requiring that the scattering amplitude is a smooth function of the energy, was recently answered affirmatively by Dean [1]. Unfortunately, there is a flaw in his reasoning, invalidating the conclusion that ambiguous sets of partial-wave amplitudes cannot intersect. In this note we show that such intersections can occur, and we give some examples.

We consider the spinless case in sect. 2. In sect. 3 we treat spin 0-spin $\frac{1}{2}$ scattering. We employ the method first introduced by Barrelet [2] to emphasize the similarity with the spinless case. Finally we gather some conclusions in sect. 4.

2. Scattering of spinless particles

If we assume that the scattering amplitude F, at an energy E, is a polynomial of degree L in the variable $z = \cos\theta$, it can be expressed in terms of its zeros $z_i(E)$ and

* On leave from State University of Groningen.

one complex constant [3]

$$F(z) = F(1) \prod_{i=1}^{L} \frac{z - z_i}{1 - z_i} \ . \tag{2.1}$$

The differential cross section

$$k^2 \frac{d\sigma}{d\Omega}(x) \equiv \sigma(x) = |F(x)|^2, \quad (-1 \leqslant x \leqslant +1) \tag{2.2}$$

remains unchanged under

(i) replacement of z_i by z_i^*,

(ii) multiplication of F by a constant phase factor exp $i\phi$, and combinations of these transformations. No measurement of $d\sigma/d\Omega$ can distinguish between two amplitudes which are related by such a transformation. The transformations (i) also leave the total cross section invariant, whereas (ii) in general do not.

New amplitudes generated by transformations (i) and (ii) are acceptable only if they satisfy the unitarity relation, which is most easily expressed in terms of the partial waves

$$f_l = \tfrac{1}{2} \int_{-1}^{+1} dx\, F(x)\, P_l(x) \, , \tag{2.3}$$

and for elastic scattering takes the form

$$\mathrm{Im}\, f_l \geqslant |f_l|^2 \, . \tag{2.4}$$

After all ambiguities at all energies have been constructed, one has to require that a possible alternative solution behaves smoothly with respect to the energy-variable, and connects smoothly with regions where the phase-shift analysis is unique (usually the energy region where there is no inelasticity and (2.4) is an equality). This implies that the alternative solution must branch off at some energy from the original solution, which can only happen if at that energy the transformation applied to the original amplitude reduces to the identity transformation. In the case of transformations (i) this implies that a zero of $F(z)$ becomes real, or, more generally, that $F(z)$ contains a factor which is a polynomial in z (of degree $\leqslant L$) with real coefficients. In such a situation continuity in energy will not suffice to resolve the ambiguity, and in fact such situations are known to occur in some physical cases.

A most striking example occurs in a recent analysis of $\pi\pi$ scattering [4], where zero trajectories pass through or come very close to the real axis (see in particular fig. 2 of ref. [4]). Other examples are known to us from our own analysis of the discrete ambiguities of $\alpha\alpha$ scattering amplitudes [5]. In sect. 3 we shall show that an analogous phenomenon can, and in fact does, happen in spin 0-spin $\frac{1}{2}$ scattering.

3. Spin 0-spin $\frac{1}{2}$ scattering

We start from the usual form for the amplitude

$$f(x) = g(x) + i\boldsymbol{\sigma}\cdot\boldsymbol{n}\, h(x)\,. \tag{3.1}$$

We first introduce a new function $k(x) = (1 - x^2)^{-1/2}h(x)$ and then define

$$\widetilde{F}(z) = g(z) + (z^2 - 1)^{1/2}k(z) \tag{3.2}$$

which is equal to the transversity amplitudes $g \pm ih$ above and below the cut from -1 to $+1$. The mapping

$$\zeta = z + (z^2 - 1)^{1/2} \tag{3.3}$$

maps the physical region into the unit circle in such a way that the upper (lower) lip of the cut $-1 \leqslant z \leqslant 1$ is mapped into $\zeta = e^{i\phi}$, $\pi \geqslant \phi \geqslant 0$ $(-\pi \leqslant \phi \leqslant 0)$. From now on we shall work with the function $F(\zeta) = \widetilde{F}(z)$.

The function

$$\tau(\zeta) = F(\zeta)F^*(1/\zeta^*) \tag{3.5}$$

takes on the values

$$\tau(e^{\pm i\theta}) = \sigma(\cos\theta)\ [1 \pm P(\cos\theta)] \tag{3.6}$$

on the unit circle, where the differential cross section σ and the polarization P are, in terms of g and h:

$$\sigma(x) = |g(x)|^2 + |h(x)|^2\,, \qquad \sigma(x)P(x) = 2\ \mathrm{Im}\,(g(x)\,h^*(x))\,. \tag{3.7}$$

For $F(\zeta)$ we make the expansion

$$F(\zeta) = \sum_{l=0}^{\infty} (l+1)\, f_{l+}P_{l+}(\zeta) + \sum_{l=1}^{\infty} l f_{l-}P_{l-}(\zeta) \tag{3.8}$$

where [2]

$$P_{l\pm}(\zeta) = P_l(z) \pm 2(z^2 - 1)^{1/2}\, P_l'(z)/(2l + 1 \pm 1)\,. \tag{3.9}$$

We must consider all transformations of the amplitude $F(\zeta)$ that leave $\tau(\zeta)$, and therefore σ and P, the same. From (3.5) it is clear that the following transformations and their products leave τ invariant

$$F(\zeta) \to -F^*(1/\zeta^*)/\zeta\,, \tag{3.10}$$

$$F(\zeta) \to \zeta^n F(\zeta)\,, \qquad n = \pm 1, \pm 2, \ldots\,. \tag{3.11}$$

Formula (3.10) is the generalized Minami ambiguity [6,7] (3.11) was first considered by Dean and Lee [8], see also [9].

If we assume that the scattering amplitude has a finite number of partial waves, we can express F in terms of its zeros $\zeta_i(E)$ and a complex constant

$$F(\zeta) = F(1)\,\zeta^{-L} \prod_{i=1}^{N} \frac{\zeta - \zeta_i}{1 - \zeta_i} \,. \tag{3.12}$$

Here L is the highest value of l occurring in (3.8), and the number of zeros N equals $2L-1$ $(2L)$ if $f_{L+} = 0$ $(f_{L+} \neq 0)$. All possible phase-shift ambiguities that involve only a finite number of partial waves now result from the transformations (3.10–11), and in addition from

(i) replacement of ζ_i by $1/\zeta_i^*$, we call this transformation T_i,

(ii) multiplication of F by a constant phase factor $\exp(i\phi)$, and any products of these transformations. Note that transformation (3.10) is the product of all T_i and a suitably chosen phase factor. As in the spinless case unitarity is not automatic. The transformations (ii) will in general change the total cross section.

If an alternative solution is to branch off at some energy from the original solution, or if two solutions intersect, the transformation applied to the original amplitude must approach the identity transformation at that particular energy. This is clearly impossible for the transformation (3.11). The transformation T_i reduces to the identity when the trajectory $\zeta_i(E)$ crosses the unit circle. In such cases $\sigma(1 \pm P)$ vanishes for one value of θ, so that either σ has a zero, or $|P| = 1$. The transformation $T_i T_j$ becomes the identity if at some energy $\zeta_i = 1/\zeta_j^*$. This can happen for ζ_i not on the physical region, so that $\sigma(1 \pm P)$ is not necessarily zero for some value of θ.

Like in the spinless case explicit examples of such ambiguities are observed, and the phenomenon of intersection of ambiguous amplitudes occurs.

As an interesting illustration we consider the amplitudes resulting from a phase-shift analysis by Wienhard et al. [10] of proton scattering by ^{12}C at 24 energies from 9.95 MeV to 10.90 MeV, neglecting the Coulomb part of the ampltiude. These amplitudes were used in a Barrelet analysis by Heemskerk et al. [11]. The zero trajectories in the ζ plane, with a part of the unit circle, are shown in fig. 1. For convenience we plot all zeros inside the unit circle replacing a zero ζ_i outside the circle by $1/\zeta_i^*$. A full line connects the zeros (represented by dots) which in the analysis of Wienhard et al. are inside the unit circle, a broken line connects the zeros (represented by crosses) outside the circle. At twelve points a trajectory crosses the circle, indicated in fig. 1 by an arrow and the corresponding energy. Whenever a trajectory crosses the lower (upper) half of the circle, the polarization at that energy becomes one (minus one) at the corresponding c.m. scattering angle $\theta = i \ln \zeta$ $(\theta = -i \ln \zeta)$.

Calculations show [11] that, when a trajectory crosses the circle, at least at two and often at many more energies neighbouring the point of crossing, the relevant zero ζ can be replaced by $1/\zeta^*$ to give an alternative unitary amplitude that fits experiment equally well (the only exception being zero nr. 3 at 10.07 MeV). It is therefore possible to construct an alternative amplitude which as a function of energy intersects once or more than once the original amplitude.

In an inelastic scattering process the unitarity constraint is much less stringent,

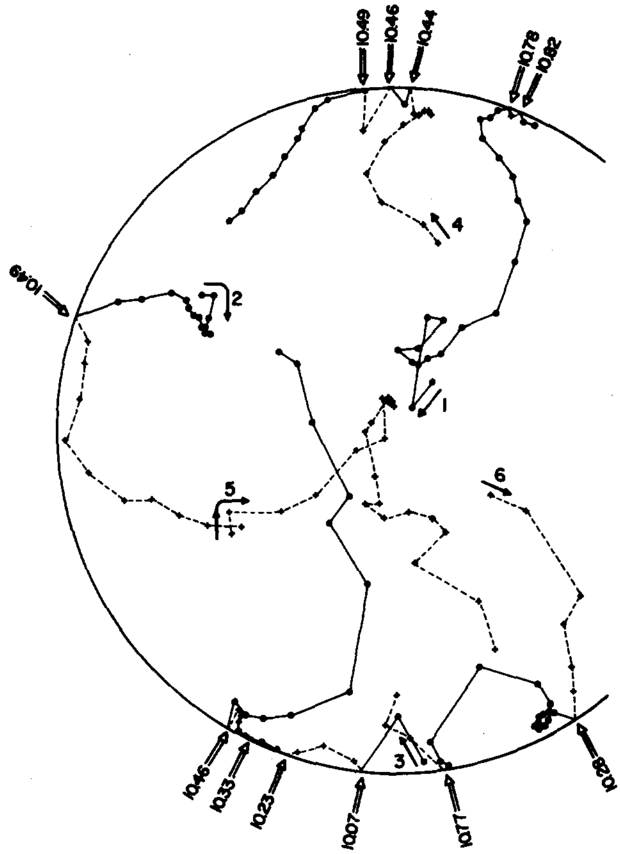

Fig. 1. Trajectories of zeros of the p-^{12}C scattering amplitude in the complex ζ plane for energies from 9.95 MeV to 10.90 MeV. The arrows indicate the points where trajectories cross the unit circle.

and takes the form

$$|f_l|^2 \leqslant \tfrac{1}{4} \; . \tag{3.13}$$

In the analysis of such a process one can therefore expect a richer ambiguity structure. An example is provided by various analyses of $K^- p \to \pi^0 \Lambda$ [9,12]. In these analyses zero trajectories frequently cross the unit circle in the ζ plane. Whenever this happens, $P \pm 1$ vanishes somewhere on the physical region. This does not imply that both g and h vanish at the same value of $\cos\theta$ (as suggested in a footnote in ref. [1]), but only that the combination $g + ih$ or $g - ih$ has a zero in the physical region.

4. Discussion

We have shown that the intersection of ambiguous sets of partial-wave amplitudes is possible for spin 0-spin $\frac{1}{2}$ scattering as well as for spin 0-spin 0 scattering. The Barrelet formalism which we employ in sect. 3 makes the analogy between the possible mechanism of intersection in these two cases quite clear. The fact that ambiguous amplitudes can intersect, implies that continuity in energy does not always resolve the non-uniqueness in phase-shift analysis.

We have considered only scattering amplitudes with a finite number of partial waves. If one allows an infinite number of partial waves, decreasing exponentially with l, one can also have, in the inelastic region, a continuum ambiguity [13]. As in this case the amplitudes can be varied continuously at each energy, the resulting ambiguities cannot be resolved by the requirement of continuity in energy alone.

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